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2002 J. Phys. A: Math. Gen. 35 2213

(http://iopscience.iop.org/0305-4470/35/9/311)

J. Phys. A: Math. Gen. **35** (2002) 2213–2221 PII: S0305-4470(02)29756-5

The Appell hypergeometric functions and classical separable mechanical systems

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Received 8 October 2001, in final form 21 January 2002
Published 22 February 2002
Online at stacks.iop.org/JPhysA/35/2213

Abstract

A relationship between two old mathematical subjects is observed: the theory of hypergeometric functions and the separability in classical mechanics. Separable potential perturbations of integrable billiard systems and the Jacobi problem for geodesics on an ellipsoid are expressed through the Appell hypergeometric functions F_4 of two variables. Even when the number of degrees of freedom increases, if an ellipsoid is symmetric, the number of variables in the hypergeometric functions does not increase. Wider classes of separable potentials are given by the obtained new formulae automatically.

PACS numbers: 02.30.Gp, 45.50.-j

1. Introduction

Appell introduced four families of hypergeometric functions of two variables in the 1880s. Soon, he applied them in a solution of the Tisserand problem in the celestial mechanics. The Appell functions have several other applications, for example in the theory of algebraic equations, algebraic surfaces.... The aim of this paper is to point out the relationship between the Appell functions F_4 and another subject from classical mechanics—separability of variables in the Hamilton–Jacobi equations.

The equation

$$\lambda V_{xy} + 3(yV_x - xV_y) + (y^2 - x^2)V_{xy} + xy(V_{xx} - V_{yy}) = 0 \tag{1}$$

appeared in Kozlov's paper [1] as a condition on the function $V = V(x, y)$ to be an integrable perturbation of certain type for billiard systems inside an ellipse

$$\frac{x^2}{A} + \frac{y^2}{B} = 1 \qquad \lambda = A - B. \tag{2}$$

This equation is a special case of the Bertrand–Darboux equation [2–4]

$$(V_{yy} - V_{xx})(-2axy - b'y - bx + c_1) + 2V_{xy}(ay^2 - ax^2 + by - b'x + c - c') + V_x(6ay + 3b) + V_y(-6ax - 3b') = 0. \tag{3}$$

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It corresponds to the choice $a = -1/2$, $b = b' = c_1 = 0$, $c - c' = -\lambda/2$. The Bertrand–Darboux equation represents the necessary and sufficient condition for a natural mechanical system with two degrees of freedom

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y)$$

to be separable in elliptical coordinates or some of their degenerations.

Solutions of equation (1) in the form of the Laurent polynomials in x, y were described in [5]. The starting observation of this paper, that such solutions are simply related to the well known hypergeometric functions of the Appell type, is presented in section 3. Such a relation automatically gives a wider class of solutions of equation (1)—new potentials are obtained for non-integer parameters. But what is more important is that it shows the existence of a connection between the separability of classical systems on the one hand, and the theory of hypergeometric functions on the other. Basic references for the Appell functions are [6–8]. Further, in section 3, similar formulae for potential perturbations for the Jacobi problem for geodesics on an ellipsoid from [9] and for billiard systems on surfaces with constant curvature from [10], are given.

In the case of more than two degrees of freedom, the natural generalization for equations (1) and (3) is the system (4). In [11], considering billiard systems inside an ellipsoid in R^3, the system (4) is derived for the case of three degrees of freedom, and its Laurent polynomial solutions are given. In section 4, we express these solutions through the hypergeometric functions for the case of the symmetric ellipsiod only. We again obtain the Appell hypergeometric functions of two variables.

The system

$$\begin{aligned}
&(a_i - a_r)^{-1}(x_i^2 V_{rs} - x_i x_r V_{is}) = (a_i - a_s)^{-1}(x_i^2 V_{rs} - x_i x_s V_{ir}) \qquad i \neq r \neq s \neq i \\
&(a_i - a_r)^{-1} x_i x_r (V_{ii} - V_{rr}) - \sum_{j \neq i,r} (a_i - a_j)^{-1} x_i x_j V_{jr} \\
&\qquad + V_{ir}\left[\sum_{j \neq i,r} (a_i - a_j)^{-1} x_j^2 + (a_r - a_i)^{-1}(x_i^2 - x_r^2)\right] + V_{ir} \\
&\qquad + 3(a_i - a_r)^{-1}(x_r V_i - x_i V_r) = 0 \qquad i \neq r
\end{aligned} \tag{4}$$

where $V_i = \partial V/\partial x_i$, of $(n-1)\binom{n}{2}$ equations was formulated in [12] for an arbitrary number of degrees of freedom n. In [12] the generalization of the Bertrand–Darboux theorem is proved. According to that theorem, the solutions of the system (4) are potentials separable in generalized elliptic coordinates (see the theorem after lemma 2, below).

Deeper explanation of the connection between the separability in elliptic coordinates and the Appell hypergeometric functions is not yet known.

2. Basic notation

The function F_4 is one of the four hypergeometric functions in two variables introduced by Appell [7, 8] and defined as a series

$$F_4(a, b, c, d; x, y) = \sum \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!}$$

where $(a)_n$ is the standard Pochhammer symbol:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\ldots(a+n-1)$$

$$(a)_0 = 1.$$

(For example $m! = (1)_m$.)

The series F_4 is convergent for $\sqrt{x} + \sqrt{y} \leqslant 1$. The functions F_4 can be analytically continued to the solutions of the equations

$$x(1-x)\frac{\partial^2 F}{\partial x^2} - y^2\frac{\partial^2 F}{\partial y^2} - 2xy\frac{\partial^2 F}{\partial x \partial y} + [c - (a+b+1)x]\frac{\partial F}{\partial x} - (a+b+1)y\frac{\partial F}{\partial y} - abF = 0$$

$$y(1-y)\frac{\partial^2 F}{\partial y^2} - x^2\frac{\partial^2 F}{\partial x^2} - 2xy\frac{\partial^2 F}{\partial x \partial y} + [c' - (a+b+1)y]\frac{\partial F}{\partial y} - (a+b+1)x\frac{\partial F}{\partial x} - abF = 0.$$

3. The separable systems with two degrees of freedom

3.1. Billiard inside an ellipse

Following [1, 5] we will start with a billiard system which describes a particle moving freely within an ellipse (2). At the boundary we assume elastic reflections with equal impact and reflection angles. This system is completely integrable and it has an additional integral

$$K_1 = \frac{\dot{x}^2}{A} + \frac{\dot{y}^2}{B} - \frac{(\dot{x}y - \dot{y}x)^2}{AB}.$$

We are interested in potential perturbation $V = V(x, y)$ such that the perturbed system has an integral $\tilde{K}_1$ of the form

$$\tilde{K}_1 = K_1 + k_1(x, y)$$

where $k_1 = k_1(x, y)$ depends only on coordinates. This specific condition leads to equation (1) on V (see [1]).

In [5] the Laurent polynomial solutions of equation (1) were given. The basic set of solutions consists of the functions

$$V_k = \sum_{i=0}^{k-2}(-1)^i \sum_{s=1}^{k-i-1} U_{kis}(x, y, \lambda) + y^{-2k} \qquad k \in N$$

$$W_k = \sum_{i=0}^{k-2} \sum_{s=1}^{k-i-1} (-1)^s U_{kis}(y, x, \lambda) + x^{-2k} \qquad k \in N$$

where

$$U_{kis} = \binom{s+i-1}{i} \frac{[1-(k-i)][2-(k-i)]\ldots[s-(k-i)]}{\lambda^{s+i} s!} x^{2s} y^{-2k+2i}.$$

Now, we rewrite the above formulae:

$$\begin{aligned} V_k &= \sum_{i=0}^{k-2}(-1)^i \sum_{s=1}^{k-i-1} U_{kis}(x, y, \lambda) + y^{-2k} \qquad k \in N \\ &= \sum_{i=0}^{k-2}(-1)^i \sum_{s=1}^{k-i-1} \frac{\Gamma(s+i)\Gamma(s+i-k+1)}{\Gamma(i+1)\Gamma(s)\Gamma(i-k+1)\Gamma(s+1)} \frac{x^{2s} y^{2(i-k)}}{\lambda^{s+i}} + y^{-2k} \\ &= \frac{1}{y^{2k}}\left((1-k)\sum_{i=0}^{k-2}\sum_{s=1}^{k-i-1} \frac{(1)_{s+i-1}(2-k)_{s+i-1}}{i!(1)_{s-1}s!(1-k)_i} \frac{x^{2s}}{\lambda^s} \frac{(-y^2)^i}{\lambda^i} + 1\right) \end{aligned}$$

$$= \frac{1}{y^{2k}}\left((1-k)\frac{x^2}{\lambda}\sum_{i=0}^{k-2}\sum_{s=0}^{k-i-2}\frac{(1)_{s+i}(2-k)_{s+i}}{(2)_s(1-k)_i}\frac{(x^2)^s}{s!\lambda^s}\frac{(-y^2)^i}{i!\lambda^i}+1\right)$$
$$= \frac{1}{\tilde{y}^k\lambda^k}((1-k)\tilde{x}F_4(1;\ 2-k;\ 2,\ 1-k,\ \tilde{x},\ -\tilde{y})+1)$$

where $\tilde{x} = x^2/\lambda,\ \tilde{y} = -y^2/\lambda$, and F_4 is the Appell function. We have just obtained a simple formula which expresses the potentials V_k, from [5], for $k \in N$ through the Appell functions. (The scalar coefficient λ^{-k} is not essential and we omit it henceforth.) We can use this formula to extend the family of solutions of equation (1) out of the set of the Laurent polynomials. We obtain new solutions of equation (1) if we let the parameter k in the last formula be arbitrary, not only a natural number.

Let $V(x, y) = \sum a_{nm}x^n y^m$. Then equation (1) reduces to

$$\lambda nma_{n,m} = (n+m)(ma_{n-2,m} - na_{n,m-2}). \tag{5}$$

If one of the indices, for example the first one, belongs to Z, then V does not have essential singularities. Put $a_{0,-2\gamma} = 1$, where γ is not necessarily an integer.

If we define

$$a_{\underbrace{2s+2}_{n},\underbrace{2i-2\gamma}_{m}} = \frac{(-1)^i(1)_{s+i}(2-\gamma)_{s+i}}{(2)_s(1-\gamma)_i s!i!\lambda^{s+i}} \tag{6}$$

it can easily be seen that (6) is a solution of equation (5). So, let us denote

$$V_\gamma = \tilde{y}^{-\gamma}((1-\gamma)\tilde{x}F_4(1, 2-\gamma, 2, 1-\gamma, \tilde{x}, \tilde{y})+1). \tag{7}$$

Then we have the following theorem.

Theorem 1. *Every function V_γ given with (7) and $\gamma \in C$ is a solution of equation (1).*

Theorem 1 gives new potentials for non-integer γ.

Mechanical interpretation: With $\gamma \in R^-$ and the coefficient multiplying V_γ positive, we have a potential barrier along the x-axis. We can consider billiard motion in the upper half-plane. Then we can assume that a cut is done along the negative part of the y-axis, in order to get a unique-valued real function as a potential.

Solutions of equation (1) are also connected with interesting geometric subjects. We briefly mention some of them at the end of section 5.

3.2. The Jacobi problem for geodesics on an ellipsoid

The Jacobi problem for the geodesics on an ellipsoid

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C} = 1$$

has an additional integral

$$K_1 = \left(\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}\right)\left(\frac{\dot{x}^2}{A}+\frac{\dot{y}^2}{B}+\frac{\dot{z}^2}{C}\right).$$

Potential perturbations $V = V(x, y, z)$ such that perturbed systems have integrals of the form

$$\tilde{K}_1 = K_1 + k(x, y, z)$$

satisfy the following system (see [9]):

$$
\begin{aligned}
&\left(\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}\right)V_{xy}\frac{A-B}{AB}-3\frac{y}{B^2}\frac{V_x}{A}+3\frac{x}{A^2}\frac{V_y}{B}+\left(\frac{x^2}{A^3}-\frac{y^2}{B^3}\right)V_{xy}\\
&\qquad+\frac{xy}{AB}\left(\frac{V_{yy}}{A}-\frac{V_{xx}}{B}\right)+\frac{zx}{CA^2}V_{zy}-\frac{zy}{CB^2}V_{zx}=0\\
&\left(\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}\right)V_{yz}\frac{B-C}{BC}-3\frac{z}{C^2}\frac{V_y}{B}+3\frac{y}{B^2}\frac{V_z}{C}+\left(\frac{y^2}{B^3}-\frac{z^2}{C^3}\right)V_{yz}\\
&\qquad+\frac{yz}{BC}\left(\frac{V_{zz}}{B}-\frac{V_{yy}}{C}\right)+\frac{xy}{AB^2}V_{xz}-\frac{xz}{AC^2}V_{xy}=0\\
&\left(\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}\right)V_{zx}\frac{C-A}{AC}-3\frac{x}{A^2}\frac{V_z}{C}+3\frac{z}{C^2}\frac{V_x}{A}+\left(\frac{z^2}{C^3}-\frac{x^2}{A^3}\right)V_{zx}\\
&\qquad+\frac{xz}{AC}\left(\frac{V_{xx}}{C}-\frac{V_{zz}}{A}\right)+\frac{zy}{BC^2}V_{xy}-\frac{yx}{BA^2}V_{yz}=0.
\end{aligned}
\tag{8}
$$

System (8) replaces equation (1) in this problem. Solutions of the system in the Laurent polynomial form were found in [9]. We can transform them in the following way:

$$
\begin{aligned}
V_{l_0}(x,y,z)&=\sum_{0\leqslant k\leqslant s,k+c\leqslant l_0}(-1)^s\binom{s+k-1}{k}(x^2)^{-l_0+k}(y^2)^s(z^2)^{l_0-(k+s)-1}\\
&\quad\times\frac{C^{s+k}(C-A)^s(C-B)^k2^{k+s}(-l_0+1)\ldots(-l_0+(k+s))}{B^kA^s(B-A)^{k+s}2^s2^ks!(-l_0+1)\ldots(-l_0+k)}(z^2)^{l_0-(k+s)-1}\\
&=\sum\frac{(s+k-1)!(-l_0+1)(-l_0+2)_{s+k-1}(z^2)^{l_0}}{k!(s-1)!s!(-l_0+1)_k(x^2)^{l_0}}\\
&\quad\times\left[\frac{x^2C(A-C)}{z^2(B-A)A}\right]^s\left[\frac{y^2C(C-B)}{z^2(B-A)B}\right]^k\\
&=(-l_0+1)\left(\frac{z^2}{x^2}\right)^{l_0}\sum\frac{(1)_{s+k-1}(-l_0+2)_{s+k-1}}{(2)_{s-1}(-l_0+1)_k}\hat{x}^s\hat{y}^k\\
&=(-l_0+1)\left(\frac{z^2}{x^2}\right)^{l_0}F_4(1;-l_0+2;2,-l_0+1,\hat{x},\hat{y})
\end{aligned}
$$

where

$$
\frac{x^2C(A-C)}{z^2(B-A)A}=\hat{x}\qquad\frac{y^2C(C-B)}{z^2(B-A)B}=\hat{y}.
$$

In the above formulae l_0 is an integer. We have the following straightforward generalization.

Theorem 2. *For every $\gamma\in C$ the function*

$$
V_\gamma=(-\gamma+1)\left(\frac{z^2}{x^2}\right)^\gamma F_4(1;-\gamma+2;2,-\gamma+1,\hat{x},\hat{y})
$$

is a solution of system (8).

3.3. Billiard systems on the constant-curvature surfaces

Potential perturbations of billiard systems on constant-curvature surfaces were analysed in [10]. Following the notation of [10], let the billiard D_S be a subset of the surface Σ_S, of the curvature $S=+1$ or -1, bounded with the quadric Q_S, where

$$
\Sigma_+=\{r=(x,y,z)\in R^3\mid\langle r,r\rangle_+=1\}\qquad\langle r_1,r_2\rangle_+=x_1y_1+x_2y_2+x_3y_3
$$

$\Sigma_- = \{r = (x, y, z) \in R^3 \mid \langle r, r\rangle_- = -1,\ z > 0\}$ $\qquad \langle r_1, r_2\rangle_- = x_1y_1 + x_2y_2 - x_3y_3$

$Q_S = \Sigma_S \cap \{r \in R^3 \mid \langle Qr, r\rangle_s = 0\} \neq \emptyset \qquad Q = \mathrm{diag}\left(\frac{1}{A}, \frac{1}{B}, \frac{1}{C}\right).$

Then the billiard system has the integral

$$K = \frac{(\dot{x}y - \dot{y}x)^2}{AB} + S\frac{(\dot{x}z - \dot{z}x)^2}{AC} + S\frac{(\dot{z}y - \dot{y}z)^2}{BC}.$$

As before, we are looking for potentials $V = V(x, y, z)$ such that the perturbed system has an integral of the form

$$\tilde{K} = K + k(x, y, z).$$

In this case the condition is given by the system [10]:

$$\begin{aligned}
&3CyV_x - 3CxV_y + V_{xy}(C(y^2 - x^2) + Kz^2(B - A))\\
&\qquad + CxyV_{xx} - CxyV_{yy} + AzyV_{zx} - BzxV_{zy} = 0\\
&3BzV_x - K3BxV_z + V_{xz}(B(z^2 - Kx^2) + Ky^2(C - A))\\
&\qquad + BzxV_{xx} - KBzxV_{zz} + AzyV_{xy} - KCyxV_{yz} = 0\\
&3AzV_y - K3AyV_z + V_{yz}(A(z^2 - Ky^2) + Kx^2(C - B))\\
&\qquad + AzyV_{yy} - KAzyV_{zz} + BzxV_{xy} - KCxyV_{xz} = 0.
\end{aligned} \tag{9}$$

Starting from the solutions from [10]

$$V_{l_0} = \frac{1}{z^{2l_0}} \sum_{\substack{0\leqslant k\leqslant l_0-1\\ 0\leqslant m\leqslant l_0-k-1}} a_{m,k} x^{2m} y^{2l_0-2-2k-2m} z^{2k}$$

where

$$a_{m,k} = K^{l_0-k-1}\left(\frac{C-B}{C-A}\right)^m \binom{l_0-k-1}{m}\binom{k+m-1}{k}\left(\frac{A-B}{C-A}\right)^k$$

we come to the following theorem.

Theorem 3. *The functions*

$$V_\gamma = \hat{y}^{-\gamma}((1-\gamma)x^2 F_4(1, 2-\gamma, 2, 1-\gamma, \hat{x}, \hat{y}) + 1)$$

where

$$\frac{x^2(B-C)}{y^2(C-A)} = \hat{x} \qquad K\frac{z^2(A-B)}{y^2(C-A)} = \hat{y}$$

are solutions of the system (9), for $\gamma \in C$.

4. More than two degrees of freedom

In the previous section, we have seen that integrable perturbations of separable systems with two degrees of freedom led to the hypergeometric functions of two variables. Now, one can expect that in a case of more than two degrees of freedom, the integrable potentials are connected with hypergeometric functions again, but with more than two variables. We will consider the billiard system inside an ellipsoid in R^3, and we will see that the corresponding potential perturbations are still related to the Appell function F_4 of two variables, if the ellipsoid is symmetric.

4.1. Billiards inside a symmetric ellipsoid in R^3

Let us consider the billiard system within an ellipsoid in R^3:

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=1.$$

The potential perturbations $W = W(x, y, z)$ of such systems in the form of Laurent polynomials were calculated in [11]. They satisfy the system (4) for $n = 3$:

$$W_{l_0}=\frac{1}{z^{2l_0}}\sum_{0\leqslant m+n+k<l_0}\frac{(l_0-k-1)!(-1)^n}{m!n!(l_0-1-k-m-n)!}\frac{P^k_{m,n}(\beta,\gamma)}{\gamma^{m+k}\beta^{n+k}}x^{2m}y^{2n}z^{2k} \tag{10}$$

where

$$P^k_{m,n}(\beta,\gamma)=\sum_{i=0}^{k}\binom{m+k-1-i}{k-i}\cdot\binom{n+i-1}{i}(-1)^i\beta^{k-i}\gamma^i$$

and $\beta = B - C,\ \gamma = C - A$.

The symmetric case $A = B$ corresponds to the condition $\gamma + \beta = 0$.

Lemma 1. *If* $\gamma + \beta = 0$ *then we have*

$$P^k_{m,n}(\beta,-\beta)=\binom{k+m+n-1}{k}\beta^k. \tag{11}$$

Proof.

$$\begin{aligned}P^k_{m,n}(\beta,-\beta)&=\beta^k\sum_{i=0}^{k}\binom{m+k-i-1}{k-i}\binom{n+i-1}{i}\\&=\beta^k\sum_{i=0}^{k}\binom{k-i+m-1}{k-i}\binom{i+n-1}{i}\\&=\binom{k+m+n-1}{k}\beta^k.\end{aligned}$$

□

By putting (11) into (10) we get, using $\gamma = -\beta$,

$$\begin{aligned}W_{l_0}&=C\frac{1}{\hat{z}^{l_0}}\sum_{m+n+k=1}^{l_0}\frac{(l_0-1)\ldots(l_0-m-n-k)(m+n+k-1)!(-1)^n}{(l_0-1)\ldots(l_0-k)(m+n-1)!}\frac{\hat{x}^m}{m!}\frac{\hat{y}^n}{n!}\frac{\hat{z}^k}{k!}\\&=\frac{C}{(\hat{z})^{l_0}}\left[\sum_{m+n+k=1}^{l_0}\frac{(-l_0+2)\ldots(-l_0+m+n+k)(1)_{m+n+k-1}}{(-l_0+1)\ldots(-l_0+k)(-1)^{-m}(1)_{m+n-1}}\frac{\hat{x}^m}{m!}\frac{\hat{y}^n}{n!}\frac{\hat{z}^k}{k!}+1\right]\end{aligned}$$

where $\hat{x}=\frac{x^2}{\gamma},\ \hat{y}=\frac{y^2}{\beta},\ \hat{z}=\frac{z^2}{\gamma}$.

$$\begin{aligned}W_{l_0}&=C\cdot\hat{z}^{-l_0}\left(\sum_{m+n+k=1}^{l_0}\frac{(1)_{m+n+k-1}(2-l_0)_{m+n+k-1}(-1)^m}{(1)_{m+n-1}(1-l_0)_k}\frac{\hat{x}^m}{m!}\frac{\hat{y}^n}{n!}\frac{\hat{z}^k}{k!}+1\right)\\&=C\cdot\hat{z}^{-l_0}\left(\sum_{m+n+k=1}^{l_0}\frac{(1)_{m+n+k-1}(2-l_0)_{m+n+k-1}(-1)^m(m+n)!}{(2)_{m+n-1}(1-l_0)_k(m+n-1)!}\frac{\hat{x}^m}{m!}\frac{\hat{y}^n}{n!}\frac{\hat{z}^k}{k!}+1\right)\\&=C\cdot\hat{z}^{-l_0}\left(\sum_{m+n+k=1}^{l_0}\frac{(1)_{m+n+k-1}(2-l_0)_{m+n+k-1}}{(2)_{m+n-1}(1-l_0)_k}\frac{(-\hat{x}+\hat{y})^{m+n}}{(m+n-1)!}\frac{\hat{z}^k}{k!}+1\right)\\&=C\cdot\hat{z}^{-l_0}\left[(-\hat{x}+\hat{y})\sum_{0\leqslant s+k<l_0}\frac{(1)_{s+k}(2-l_0)_{s+k}}{(2)_s(1-l_0)_k}\frac{(-\hat{x}+\hat{y})^s}{s!}\frac{\hat{z}^k}{k!}+1\right]\end{aligned}$$

where $C = (1 - l_0)\gamma^{-l_0}$, $m + n - 1 = s$.

Theorem 4. *Generalizations of the integrable potential perturbations from [11] in the symmetric case are given by*

$$W_{l_0} = (\hat{z})^{-l_0}[(-\hat{x} + \hat{y})F_4(1, 2 - l_0; 2, 1 - l_0; -\hat{x} + \hat{y}, \hat{z}) + 1],$$

where $l_0 \in C$.

4.2. The case of general dimension

We consider the billiard system in R^n within an ellipsoid

$$\frac{x_1^2}{a_1} + \cdots + \frac{x_n^2}{a_n} = 1.$$

For $n \geqslant 3$ we start from a separable system with n integrals $K_1, \ldots, K_n$ which are mutually in involution, where $K_n = H$ is the Hamiltonian and

$$K_i = \sum_{j \neq i} \frac{(\dot{x}_i x_j - \dot{x}_j x_i)^2}{a_i - a_j} \qquad i = 1, \ldots, n - 1. \tag{12}$$

Then we are interested in potential perturbations $k_1, \ldots, k_n$, where $k_i = k_i(x_1, \ldots, x_n)$ depend only on coordinates and $V = k_n$. The conditions

$$\{\tilde{K}_n, \tilde{K}_i\} = 0 \qquad i = 1, \ldots, n - 1 \tag{13}$$

where

$$\tilde{K}_i = K_i + k_i \qquad i = 1, \ldots, n \tag{14}$$

are equivalent to the system (4). A nontrivial (and possibly unexpected) fact is that the new integrals commute between themselves.

Lemma 2. *From (12) and (13), it follows that*

$$\{\tilde{K}_j, \tilde{K}_i\} = 0 \qquad i, j = 1, \ldots, n - 1.$$

This was checked by direct calculation for $n = 3$ in [11]. We will give the proof in the general case after the formulation of the generalized Bertrand–Darboux theorem proved in [12].

Generalized Bertrand–Darboux theorem [12]. *For a natural Hamiltonian system with a Hamiltonian*

$$H = \tfrac{1}{2} \sum_{i=1}^{n} p_i^2 + V(x)$$

the following three conditions are equivalent:

(a) It has $n - 1$ *global, independent, involutive integrals of the form (14).*

(b) The potential V *satisfies the system (4).*

(c) The Hamilton–Jacobi equation for H *is separable in generalized elliptic coordinates* $(u_1, \ldots, u_n)$ *given by*

$$1 + \sum_{i=1}^{n} \frac{x_i}{z - a_i} = \frac{\prod_{j=1}^{n}(z - u_j)}{\prod_{k=1}^{n}(z - \alpha_k)}.$$

Proof of lemma 2. The conditions (13) for the functions defined by (12), (14) lead to the system (4). So the condition (b) of the last theorem is satisfied. The chain of implications from (b) via (c) to (a) from that theorem ends the proof. □

However, the perturbations in the general case are not connected with the Appell hypergeometric functions, except in the degenerate completely symmetric case $a_1 = \cdots = a_{n-1}$, which is an obvious generalization of theorem 4.

5. Conclusion

If we denote in (10) $-\beta/\gamma$ by q, then theorem 4 shows that the potentials W_{l_0} are certain deformations of the (Appell) hypergeometric functions. The analysis of this deformation and comparison to the known q-deformations and multivariable generalizations of the hypergeometric functions [6] remains as an interesting problem.

Potential perturbations of classical nonholonomic rigid-body problems are described in [13]. It seems that they are also connected with the hypergeometric functions.

From the geometric point of view, it is well known that billiard systems within an ellipse are closely related to the Poncelet and the Cayley theorems [14–16]. So, the Appell hypergeometric functions define natural deformations of these classical projective geometry settings.

Geometry of separable systems and their relationship to bi-Hamiltonian systems is extensively studied by several authors. Let as mention Benenti, Magri and their collaborators [17–19]. In that sense, equation (1) can be considered as a condition for the second structure to be the Poisson bracket.

Acknowledgments

This research was partially done during the author's visit to the Mathematical Department of Kyoto University as a Matsumae International Foundation Fellow. It is a great pleasure to thank Professor M Jimbo for warm hospitality and A Yu Orlov for valuable discussions. The referee's comments led to improvement of the manuscript.

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